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射影空間の超曲面と直線の幾何と Hodge 構造

The geometry of hypersurfaces and lines in projective spaces and Hodge structure

池田 京司

Atsushi Ikeda

1 Abstract

We denote by $\mathbf{G}$ the set of all lines in a complex projective space $\mathbf{P}^n$. For a hypersurface X in $\mathbf{P}^n$, the set

$$Y_m = \{(p, L) \in \mathbf{P}^n \times \mathbf{G} \mid L \text{ and } X \text{ intersect at a point } p \text{ with the multiplicity} \geq m\}$$

form a projective variety, whose defining equations are given by using the higher derivative of the defining equation of X (Proposition 2.1). The projective variety Y_m is smooth for a general hypersurface X (Theorem 2.2). A purpose of the research is to characterize some geometric properties of X by using the Hodge structure of Y_m. In this report, we give a method to describe the Hodge cohomology of Y_m by Jacobian rings, which is a new generalization of the theory of Jacobian ring for a hypersurface in $\mathbf{P}^n$ by Griffiths [2].

2 Varieties of intersection

We denote by $\mathbf{P} = \mathrm{Grass}\,(V, n)$ the Grassmannian variety of all n-dimensional subspaces in $V = H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$, and denote by $\mathcal{S}_\mathbf{P}$ (resp. $\mathcal{Q}_\mathbf{P}$) the universal sub (resp. quotient) bundle on $\mathbf{P}$. We have an exact sequence

$$0 \longrightarrow \mathcal{S}_\mathbf{P} \longrightarrow \mathcal{O}_\mathbf{P} \otimes V \longrightarrow \mathcal{Q}_\mathbf{P} \longrightarrow 0.$$

Then $\mathbf{P}$ is naturally identified with $\mathbf{P}^n$, and $\mathcal{Q}_\mathbf{P}$ is identified with $\mathcal{O}_{\mathbf{P}^n}(1)$. We denote by $\mathbf{G} = \mathrm{Grass}\,(V, n-1)$ the Grassmannian variety of all $(n-1)$-dimensional subspaces in V, and denote by $\mathcal{S}_\mathbf{G}$ (resp. $\mathcal{Q}_\mathbf{G}$) the universal sub (resp. quotient) bundle on $\mathbf{G}$. We have an exact sequence

$$0 \longrightarrow \mathcal{S}_\mathbf{G} \longrightarrow \mathcal{O}_\mathbf{G} \otimes V \longrightarrow \mathcal{Q}_\mathbf{G} \longrightarrow 0.$$

We remark that a point in $\mathbf{G}$ corresponds to a line in $\mathbf{P}^n$. We denote by Γ the subvariety of $\mathbf{P} \times \mathbf{G}$ defined as the zeros of the composition

$$p_2^* \mathcal{S}_{\mathbf{G}} \longrightarrow \mathcal{O}_{\mathbf{P}\times\mathbf{G}} \otimes V \longrightarrow p_1^* \mathcal{Q}_{\mathbf{P}},$$

where $p_1 : \mathbf{P} \times \mathbf{G} \to \mathbf{P}$ (resp. $p_2 : \mathbf{P} \times \mathbf{G} \to \mathbf{G}$) denotes the first (resp. second) projection. Then Γ is the flag variety of all pairs of a line in $\mathbf{P}^n$ and a point on the line. By the first projection $\phi = p_1|_\Gamma$, the subvariety Γ is considered as the Grassmannian bundle ($\mathbf{P}^{n-1}$-bundle)

$$\phi : \Gamma = \mathrm{Grass}\,(\mathcal{S}_{\mathbf{P}}, n-1) \longrightarrow \mathbf{P}.$$

By the second projection $\pi = p_2|_\Gamma$, the subvariety Γ is considered as the Grassmannian bundle ($\mathbf{P}^1$-bundle)

$$\pi : \Gamma = \mathrm{Grass}\,(\mathcal{Q}_{\mathbf{G}}, 1) \longrightarrow \mathbf{G}.$$

We denote by $\mathcal{Q}_\phi$ the universal quotient bundle of the Grassmannian bundle ϕ. We have exact sequences

$$0 \longrightarrow \pi^* \mathcal{S}_{\mathbf{G}} \longrightarrow \phi^* \mathcal{S}_{\mathbf{P}} \longrightarrow \mathcal{Q}_\phi \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{Q}_\phi \longrightarrow \pi^* \mathcal{Q}_{\mathbf{G}} \longrightarrow \phi^* \mathcal{Q}_{\mathbf{P}} \longrightarrow 0.$$

We introduce descending filtration

$$\mathrm{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} = \mathrm{Fil}^0 \mathrm{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} \supset \cdots \supset \mathrm{Fil}^{d+1} \mathrm{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} = 0$$

on the d-th symmetric product of $\pi^* \mathcal{Q}_{\mathbf{G}}$, where the subspace $\mathrm{Fil}^m \mathrm{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}}$ is defined as the image of the natural homomorphism

$$\mathrm{Sym}^m \mathcal{Q}_\phi \otimes \mathrm{Sym}^{d-m} \pi^* \mathcal{Q}_{\mathbf{G}} \longrightarrow \mathrm{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}}.$$

For $F \in \mathrm{Sym}^d V$, we denote by X_F the hypersurface in $\mathbf{P}$ defined as the zeros of the image of F by the natural isomorphism

$$\mathrm{Sym}^d V \simeq H^0(\mathbf{P}, \mathrm{Sym}^d \mathcal{Q}_{\mathbf{P}}),$$

we denote by $Y_{F,m}$ the subvariety in Γ defined as the zeros of the image of F by the natural homomorphism

$$\mathrm{Sym}^d V \simeq H^0(\Gamma, \mathrm{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}}) \longrightarrow H^0(\Gamma, \mathrm{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} / \mathrm{Fil}^m \mathrm{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}}),$$

and we denote by Z_F the subvariety in $\mathbf{G}$ defined as the zeros of the image of F by the natural isomorphism

$$\mathrm{Sym}^d V \simeq H^0(\mathbf{G}, \mathrm{Sym}^d \mathcal{Q}_{\mathbf{G}}).$$

Then a point in Z_F is corresponds to a line which is contained in X_F. Let L be a line in $\mathbf{P}^n$, and let p be a point on L. The fiber of the line bundle $\mathcal{Q}_\phi$ at the point on Γ corresponding to the pair (p, L) is naturally identified with the kernel of the restriction

$$H^0(L, \mathcal{O}_{\mathbf{P}^n}(1)|_L) \longrightarrow H^0(p, \mathcal{O}_{\mathbf{P}^n}(1)|_p).$$

Hence, L and X_F intersect at p with the multiplicity $\geq m$ if and only if the pair (p, L) represent a point in $Y_{F,m}$. We have a diagram

$$\begin{array}{ccccccc}
 & & \mathbf{P} & \xleftarrow{\phi} & \Gamma & \xrightarrow{\pi} & \mathbf{G} \\
 & & \cup & & \cup & & \cup \\
X_F & = & \phi(Y_{F,1}) & \longleftarrow & Y_{F,1} & \longrightarrow & \pi(Y_{F,1}) \\
 & & \cup & & \cup & & \cup \\
 & & \vdots & & \vdots & & \vdots \\
 & & \cup & & \cup & & \cup \\
 & & \phi(Y_{F,d}) & \longleftarrow & Y_{F,d} & \longrightarrow & \pi(Y_{F,d}) \\
 & & \cup & & \cup & & \cup \\
 & & \phi(Y_{F,d+1}) & \longleftarrow & Y_{F,d+1} & \longrightarrow & \pi(Y_{F,d+1}) & = & Z_F.
\end{array}$$

The morphism $\pi|_{Y_{F,m}} : Y_{F,m} \to \pi(Y_{F,m})$ is finite for $1 \leq m \leq d$, and the morphism $\pi|_{Y_{F,d+1}} : Y_{F,d+1} \to Z_F$ is a $\mathbf{P}^1$-bundle.

We remark that the isomorphism

$$\operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} / \operatorname{Fil}^m \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} \simeq \operatorname{Sym}^{d-m+1} \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \operatorname{Sym}^{m-1} \pi^* \mathcal{Q}_{\mathbf{G}}$$

is induced by the homomorphism

$$\begin{array}{rl}
H^0(L, \mathcal{O}_{\mathbf{P}^n}(d)|_L) \longrightarrow & H^0(p, \mathcal{O}_{\mathbf{P}^n}(d-m+1)|_p) \otimes H^0(L, \mathcal{O}_{\mathbf{P}^n}(m-1)|_L); \\
A_1 \cdots A_d \quad \longmapsto & \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} (A_{\sigma(1)} \cdots A_{\sigma(d-m+1)})|_p \otimes A_{\sigma(d-m+2)} \cdots A_{\sigma(d)}
\end{array}$$

where $A_i \in H^0(L, \mathcal{O}_{\mathbf{P}^n}(1)|_L)$, and $\mathfrak{S}_d$ denotes the permutation group of the index set $\{1, \ldots, d\}$.

Let $x_0, \ldots, x_n$ be a basis of V.

Proposition 2.1. *The subvariety $Y_{F,m}$ in Γ is defined as the zeros of the section*

$$\sum_{0 \leq i_1, \ldots, i_{m-1} \leq n} \frac{\partial^{m-1} F}{\partial x_{i_1} \cdots \partial x_{i_{m-1}}} \otimes x_{i_1} \cdots x_{i_{m-1}}$$
$$\in \operatorname{Sym}^{d-m+1} V \otimes \operatorname{Sym}^{m-1} V \simeq H^0(\Gamma, \operatorname{Sym}^{d-m+1} \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \operatorname{Sym}^{m-1} \pi^* \mathcal{Q}_{\mathbf{G}}).$$

The following theorem is proved by the similar way as [1], in which the corresponding results for the variety Z_F is proved.

Theorem 2.2. *Assume* $1 \leq m \leq d+1$.

1. *If* $m \leq n-1$, *then* $Y_{F,m}$ *is connected for any* $F \in \mathrm{Sym}^d V$.
2. *If* $m \leq 2n-1$, *then* $Y_{F,m}$ *is smooth of dimension* $2n-m-1$ *for general* $F \in \mathrm{Sym}^d V$.

If $Y_{F,m}$ is smooth of dimension $2n-m-1$, then we can compute some topological invariants of $Y_{F,m}$. For example, if $m = d = 2n-1$, then $\dim Y_{F,m} = 0$, and we can compute the number of the point of $Y_{F,m}$ by Schubert calculus;

$$\begin{cases} m = d = 2n-1 = 1 & \Longrightarrow \sharp Y_{F,m} = 1. \\ m = d = 2n-1 = 3 & \Longrightarrow \sharp Y_{F,m} = 9, \\ m = d = 2n-1 = 5 & \Longrightarrow \sharp Y_{F,m} = 575, \\ m = d = 2n-1 = 7 & \Longrightarrow \sharp Y_{F,m} = 99715, \\ & \quad \dots \end{cases}$$

for general F. It is similar to the case when $\dim Z_F = 0$;

$$\begin{cases} d = 2n-3 = 1 & \Longrightarrow \sharp Z_F = 1. \\ d = 2n-3 = 3 & \Longrightarrow \sharp Z_F = 9 \times 3 = 27, \\ d = 2n-3 = 5 & \Longrightarrow \sharp Z_F = 575 \times 5 = 2785, \\ d = 2n-3 = 7 & \Longrightarrow \sharp Z_F = 99715 \times 7 = 698005, \\ & \quad \dots \end{cases}$$

for general F. When $\dim Y_{F,m} = 1$ and $Y_{F,m}$ is connected, we can compute the genus of $Y_{F,m}$ for general F.

3 Jacobian rings

We denote by

$$S = \mathbf{C}[x_0, \dots, x_n, z_0, \dots, z_n] = \bigoplus_{p,q \in \mathbf{Z}} S^{p,q}$$

the polynomial ring bi-graded by $\deg x_i = (1,0)$ and $\deg z_j = (0,1)$. We define homomorphisms δ and ε by

$$\delta : S^{p,q} \longrightarrow S^{p+1,q-1}; \; A \mapsto \sum_{i=0}^{n} \frac{\partial A}{\partial z_i} \cdot x_i$$

and

$$\varepsilon : S^{p,q} \longrightarrow S^{p-1,q+1}; \; A \mapsto \sum_{i=0}^{n} \frac{\partial A}{\partial x_i} \cdot z_i.$$

For $F \in \operatorname{Sym}^d V$, we have a bi-homogeneous polynomial $F_0 \in S^{d,0}$ by considering $x_0, \dots, x_n$ as a basis of V. We set the bi-homogeneous polynomial F_k by

$$F_k = \varepsilon^k(F_0) \in S^{d-k,k}$$

for $k \geq 1$. We define the bi-graded ring $S_{F,m}$ by

$$S_{F,m} = S/(F_k;\ 0 \leq k \leq m-1),$$

and we define the bi-graded ring $R_{F,m}$ by

$$R_{F,m} = S_{F,m-1} \Big/ \Big(\frac{\partial F_{m-1}}{\partial x_i} \cdot x_j + (m-1)\frac{\partial F_{m-2}}{\partial x_i} \cdot z_j;\ 0 \leq i \leq n,\ 0 \leq j \leq n \Big)$$

for $m \geq 1$, where we set $S_{F,0} = S$. Since

$$\frac{1}{d} \sum_{i=0}^{n} \Big(\frac{\partial F_{m-1}}{\partial x_i} \cdot x_i + (m-1)\frac{\partial F_{m-2}}{\partial x_i} \cdot z_i \Big) = F_{m-1},$$

the Jacobian ring $R_{F,m}$ is a quotient ring of $S_{F,m}$.

In the following, we describe the relation between the rings $S_{F,m}$ and $R_{F,m}$ and the variety $Y_{F,m}$. Since the normal bundle $\mathcal{N}_{Y_{F,m}/\Gamma}$ of $Y_{F,m}$ in Γ is isomorphic to $(\phi^* \operatorname{Sym}^{d-m+1} \mathcal{Q}_{\mathbf{P}} \otimes \pi^* \operatorname{Sym}^{m-1} \mathcal{Q}_{\mathbf{G}})|_{Y_{F,m}}$, using Lemma 4.5, we have the following proposition.

Proposition 3.1. *If $Y_{F,m}$ is smooth of dimension $2n-m-1$, then*

$$H^0(Y_{F,m}, \mathcal{N}_{Y_{F,m}/\Gamma}) \simeq S_{F,m}^{d-m+1,m-1}$$

for $1 \leq m \leq n-1$.

We denote by T_Γ (resp. $T_{Y_{F,m}}$) the tangent bundle of Γ (resp. $Y_{F,m}$). Then we have the exact sequences

$$0 \longrightarrow \mathcal{O}_\Gamma \longrightarrow \phi^* \mathcal{S}_{\mathbf{P}}^\vee \otimes \pi^* \mathcal{Q}_{\mathbf{G}} \longrightarrow T_\Gamma \longrightarrow 0$$

and

$$0 \longrightarrow T_{Y_{F,m}} \longrightarrow T_\Gamma|_{Y_{F,m}} \longrightarrow \mathcal{N}_{Y_{F,m}/\Gamma} \longrightarrow 0,$$

where $\mathcal{S}_{\mathbf{P}}^\vee$ denotes the dual of $\mathcal{S}_{\mathbf{P}}$. We remark that the composition

$$V^\vee \otimes \pi^* \mathcal{Q}_{\mathbf{G}} \longrightarrow \phi^* \mathcal{S}_{\mathbf{P}}^\vee \otimes \pi^* \mathcal{Q}_{\mathbf{G}} \longrightarrow T_\Gamma \longrightarrow \mathcal{N}_{Y_{F,m}/\Gamma}$$

induces the homomorphism

$$\begin{array}{ccc} V^\vee \otimes V \simeq V^\vee \otimes H^0(\Gamma, \pi^* \mathcal{Q}_{\mathbf{G}}) & \longrightarrow & H^0(Y_{F,m}, \mathcal{N}_{Y_{F,m}/\Gamma}) \simeq S_{F,m}^{d-m+1,m-1}; \\ x_i^* \otimes x_j & \longmapsto & -\frac{(d-m+1)!}{d!} \big(\frac{\partial F_{m-1}}{\partial x_i} \cdot x_j + (m-1)\frac{\partial F_{m-2}}{\partial x_i} \cdot z_j \big), \end{array}$$

where $x_0^*, \dots, x_n^*$ denotes the dual basis of $x_0, \dots, x_n$. Using Lemma 4.6, we have the following theorem.

Theorem 3.2. *If $Y_{F,m}$ is smooth of dimension $2n-m-1$, then there is a natural injective homomorphism*

$$\rho : R_{F,m}^{d-m+1,m-1} \longrightarrow H^1(Y_{F,m}, T_{Y_{F,m}}),$$

and it is an isomorphism for $m \leq n-3$.

We set the integers $\alpha(n,m,d,q)$ and $\beta(n,m,d,q)$ by

$$\begin{cases} \alpha(n,m,d,q) = md - \frac{m(m-1)}{2} - n - 2 + q(d-m+1), \\ \beta(n,m,d,q) = \frac{m(m-1)}{2} - n + q(m-1). \end{cases}$$

Since the canonical bundle $\Omega_{Y_{F,m}}^{2n-m-1}$ of $Y_{F,m}$ is isomorphic to $(\phi^* \operatorname{Sym}^{\alpha(n,m,d,0)} \mathcal{Q}_{\mathbf{P}} \otimes \operatorname{Sym}^{\beta(n,m,d,0)} \mathcal{Q}_\phi)|_{Y_{F,m}}$, using Lemma 4.5, we have the following theorem.

Theorem 3.3. *If $Y_{F,m}$ is smooth of dimension $2n-m-1$, then there is a natural injective homomorphism*

$$\gamma_0 : \operatorname{Ker}(S_{F,m}^{\alpha(n,m,d,0),\beta(n,m,d,0)} \xrightarrow{\delta} S_{F,m}^{\alpha(n,m,d,0)+1,\beta(n,m,d,0)-1}) \longrightarrow H^0(Y_{F,m}, \Omega_{Y_{F,m}}^{2n-m-1}),$$

and it is an isomorphism for $m \leq n-2$ or $m = n-1 \leq 5$.

Here we remark that

$$S_{F,m}^{\alpha(n,m,d,0),\beta(n,m,d,0)} = \operatorname{Ker}(S_{F,m}^{\alpha(n,m,d,0),\beta(n,m,d,0)} \xrightarrow{\delta} S_{F,m}^{\alpha(n,m,d,0)+1,\beta(n,m,d,0)-1})$$

for $\frac{m(m-1)}{2} \leq n$.

The following theorem is proved by the similar way as Theorem 3.2, by using the exact sequence

$$0 \longrightarrow \Omega_{Y_{F,m}}^{2n-m-2} \longrightarrow T_\Gamma|_{Y_{F,m}} \otimes \Omega_{Y_{F,m}}^{2n-m-1} \longrightarrow \mathcal{N}_{Y_{F,m}/\Gamma} \otimes \Omega_{Y_{F,m}}^{2n-m-1} \longrightarrow 0.$$

Theorem 3.4. *If $\frac{m(m-1)}{2} \leq n$, and $Y_{F,m}$ is smooth of dimension $2n-m-1$, then there is a natural injective homomorphism*

$$\gamma_1 : R_{F,m}^{\alpha(n,m,d,1),\beta(n,m,d,1)} \longrightarrow H^1(Y_{F,m}, \Omega_{Y_{F,m}}^{2n-m-2}),$$

and it is an isomorphism for $m \leq n-3$.

Theorem 3.5. *If $\frac{m(m-1)}{2} \leq n$, and $Y_{F,m}$ is smooth of dimension $2n-m-1$, then the diagram*

$$\begin{array}{ccc} R_{F,m}^{d-m+1,m-1} \otimes S_{F,m}^{\alpha(n,m,d,0),\beta(n,m,d,0)} & \xrightarrow{\mu} & R_{F,m}^{\alpha(n,m,d,1),\beta(n,m,d,1)} \\ \downarrow \rho \otimes \gamma_0 & & \downarrow \gamma_1 \\ H^1(Y_{F,m}, T_{Y_{F,m}}) \otimes H^0(Y_{F,m}, \Omega_{Y_{F,m}}^{2n-m-1}) & \xrightarrow{\nu} & H^1(Y_{F,m}, \Omega_{Y_{F,m}}^{2n-m-2}) \end{array}$$

commutes, where the homomorphism μ is defined by the multiplication of the ring $R_{F,m}$, and the homomorphism ν is defined by the composition of the cup product and the contraction.

4 Calculation of cohomology

In this section, we enumerate several lemmas, which is used in the proof of theorems in Section 3. For simplicity of notations, we set the invertible sheaf $\mathcal{O}_\Gamma(p,q)$ on Γ by

$$\mathcal{O}_\Gamma(p,q) = \begin{cases} \operatorname{Sym}^p \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \operatorname{Sym}^q \mathcal{Q}_\phi & (p \geq 0,\ q \geq 0), \\ \operatorname{Sym}^p \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \operatorname{Sym}^{-q} \mathcal{Q}_\phi^\vee & (p \geq 0,\ q < 0), \\ \operatorname{Sym}^{-p} \phi^* \mathcal{Q}_{\mathbf{P}}^\vee \otimes \operatorname{Sym}^q \mathcal{Q}_\phi & (p < 0,\ q \geq 0), \\ \operatorname{Sym}^{-p} \phi^* \mathcal{Q}_{\mathbf{P}}^\vee \otimes \operatorname{Sym}^{-q} \mathcal{Q}_\phi^\vee & (p < 0,\ q < 0), \end{cases}$$

and we set $Q_{\mathbf{G}}^r = \pi^* \operatorname{Sym}^r \mathcal{Q}_{\mathbf{G}}$ for $r \geq 0$. For a sheaf $\mathcal{E}$ of $\mathcal{O}_\Gamma$-modules, we set $\mathcal{E}(p,q) = \mathcal{E} \otimes \mathcal{O}_\Gamma(p,q)$.

Lemma 4.1. *Assume $r \geq 0$. $H^0(\Gamma, Q_{\mathbf{G}}^r(p,q)) = \operatorname{Ker}(\delta^{r+1} : S^{p,q+r} \to S^{p+r+1,q-1})$.*

Lemma 4.2. *Assume $q \leq 0$ and $r \geq 0$.*

1. *$H^j(\Gamma, Q_{\mathbf{G}}^r(p,q)) = 0$ for $1 \leq j \leq n-2$.*
2. *When $n \geq 2$, if $q \geq -n+1$ or $p+r \leq -2$, then $H^{n-1}(\Gamma, Q_{\mathbf{G}}^r(p,q)) = 0$.*

Lemma 4.3. *Assume $q \leq 0$.*

1. *$H^j(\Gamma, T_\Gamma(p,q)) = 0$ for $1 \leq j \leq n-3$.*
2. *When $n \geq 3$, if $q \geq -n+1$ or $p \leq -2$, then $H^{n-2}(\Gamma, T_\Gamma(p,q)) = 0$.*

Lemma 4.4. *Assume $q \leq 0$ and $r \geq 0$.*

1. *$H^1(Y_{F,m}, Q_{\mathbf{G}}^r(p,q)|_{Y_{F,m}}) = 0$ for $1 \leq m \leq n-3$.*
2. *If $q \geq \frac{n(n-7)}{2} + 4$ or $p+r \leq (n-2)d - \frac{n(n-5)}{2} - 5$, then*

$$H^1(Y_{F,n-2}, Q_{\mathbf{G}}^r(p,q)|_{Y_{F,n-2}}) = 0.$$

Lemma 4.5. *Assume $r \geq 0$.*

1. $$H^0(Y_{F,m}, Q_{\mathbf{G}}^r(p,q)|_{Y_{F,m}}) \simeq \operatorname{Ker}(\delta^{r+1} : S_{F,m}^{p,q+r} \to S_{F,m}^{p+r+1,q-1})$$

   for $1 \leq m \leq n-2$.
2. *If $\min\{q,0\} \geq \frac{n(n-5)}{2} + 2$ or $p+r+\max\{q,0\} \leq (n-1)d - \frac{n(n-3)}{2} - 3$, then*

$$H^0(Y_{F,n-1}, Q_{\mathbf{G}}^r(p,q)|_{Y_{F,n-1}}) \simeq \operatorname{Ker}(\delta^{r+1} : S_{F,n-1}^{p,q+r} \to S_{F,n-1}^{p+r+1,q-1}).$$

Lemma 4.6. *Assume $q \leq 0$.*

1. $H^1(Y_{F,m}, T_\Gamma(p,q)|_{Y_{F,m}}) = 0$ *for* $1 \leq m \leq n-4$.
2. *If* $q \geq \frac{n(n-9)}{2} + 7$ *or* $p \leq (n-3)d - \frac{n(n-7)}{2} - 8$, *then*

$$H^1(Y_{F,n-3}, T_\Gamma(p,q)|_{Y_{F,n-3}}) = 0.$$

5 The case $n = 3$ and $m = 3$

In this section, we consider a hypersurface X_F in $\mathbf{P}^3$.

Proposition 5.1. *If the variety $Y_{F,3}$ is smooth of dimension 2, then the morphism $\phi|_{Y_{F,3}} : Y_{F,3} \to X_F$ is the double covering branched along B_F, where B_F is the divisor on X_F defined by the equation*

$$\det\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)_{0 \leq i,j \leq 3} = 0.$$

By the results in Section 3, we have natural injective homomorphisms

$$\rho : R_{F,3}^{d-2,2} \longrightarrow H^1(Y_{F,3}, T_{Y_{F,3}}),$$

$$\gamma_0 : S_{F,3}^{3d-8,0} \longrightarrow H^0(Y_{F,3}, \Omega^2_{Y_{F,3}})$$

and

$$\gamma_1 : R_{F,3}^{4d-10,2} \longrightarrow H^1(Y_{F,3}, \Omega^1_{Y_{F,3}}).$$

Proposition 5.2. *If $d \geq 3$, then the homomorphism*

$$R_{F,1}^{d,0} \xrightarrow{\varepsilon^2} R_{F,3}^{d-2,2} \longrightarrow \mathrm{Hom}_{\mathbf{C}}\,(S_{F,3}^{3d-8,0}, R_{F,3}^{4d-10,2})$$

is injective for general $F \in S^{d,0}$.

We consider the period map

$$\psi : M \longrightarrow W;\ [X_F] \longmapsto [H^2(Y_{F,3})],$$

where M denotes the set of isomorphism classes of hypersurfaces X_F in $\mathbf{P}^3$ such that $Y_{F,3}$ is smooth, and W denotes the set of isomorphism classes of Hodge structures of weight 2. By Proposition 5.2 and Theorem 3.4, the differential $d\psi$ of the period map ψ at a general point in M is injective, where we remark that the sets M and W have geometric structure. Now we have a natural question of Torelli type.

Question 5.3. For smooth surfaces X_{F_1} and X_{F_2} in $\mathbf{P}^3$, if there is an isomorphism $H^2(Y_{F_1,3}) \simeq H^2(Y_{F_2,3})$ as Hodge structures, then is there an isomorphism $X_{F_1} \simeq X_{F_2}$ as algebraic varieties?

5.1 The case $d = 3$

In the following, we assume that $d = 3$. If $Y_{F,3}$ is smooth, then $Y_{F,3}$ is a minimal algebraic surface with the geometric genus $p_g = 4$, the irregularity $q = 0$ and the square of the first chern class $c_1^2 = 6$. Such algebraic surfaces are classified by Horikawa, and $Y_{F,3}$ is called of type Ib in [3].

Proposition 5.4. *For $F \in S^{3,0}$, the cubic surface X_F is smooth if and only if $Y_{F,3}$ is a smooth surface.*

If X_F is a smooth cubic surface, then X_F contains 27 lines, which means that $\sharp Z_F = 27$. Hence $Y_{F,4}$ is a disjoint union of 27 rational curves, which are (-3)-curves in $Y_{F,3}$.

Proposition 5.5. *If X_F is a smooth cubic surface, then B_F has at most nodes as its singularities. A point $p \in X_F$ is a node of B_F if and only if there are three lines in X_F which contains the point p.*

Since the morphism $\phi|_{Y_{F,3}} : Y_{F,3} \to \mathbf{P}^3$ is the canonical map for $d = 3$, we have the following proposition.

Proposition 5.6. *For smooth cubic surfaces X_{F_1} and X_{F_2}, there is an isomorphism $X_{F_1} \simeq X_{F_2}$ if and only if there is an isomorphism $Y_{F_1,3} \simeq Y_{F_2,3}$.*

In the case when $d = 3$, the Hodge structure $H^2(X_F)$ is trivial, but the Hodge structure $H^2(Y_{F,3})$ is not trivial. Hence the Question 5.3 is particularly interesting in this case.

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GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY,
TOYONAKA, OSAKA, 560-0043, JAPAN
E-mail address: atsushi@math.sci.osaka-u.ac.jp